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A Geometrical Theory of Spin Motion

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A discussion of the fundamental interrelation of geometry and physical laws with Lie groups leads to a reformulation and heuristic modification of the principle of inertia and the principle of equivalence, which is based on the simple de Sitter group instead of the Poincaré group. The resulting law of motion allows a unified formulation for structureless and spinning test particles. A metrical theory of gravitation is constructed with the modified principle, which is structured after the geometry of the manifold of the de Sitter group. The theory is equivalent to a particular Kaluza–Klein theory in ten dimensions with the Lorentz group as gauge group. A restricted version of this theory excludes torsion. It is shown by a reformulation of the energy momentum complex that this version is equivalent to general relativity with a cosmologic term quadratic in the curvature tensor and in which the existence of spinning particle fields is inherent from first principles. The equations of the general theory with torsion are presented and it is shown in a special case how the boundary conditions for the torsion degree of freedom have to be chosen such as to treat orbital and spin angular momenta on an equal footing. The possibility of verification of the resulting anomalous spin–spin interaction is mentioned and a model imposed by the group topology of $SO(3,2)$ is outlined in which the unexplained discrepancy between the magnitude of the discrete valued coupling constants and the gravitational constant in Kaluza–Klein theories is resolved by the identification of identical fermions as one orbit. The mathematical structure can be adapted to larger groups to include other degrees of freedom.

1. INTRODUCTION

After the first non-Euclidian geometries were introduced in the 19th century, Riemann (1976) pointed out the complementarity of geometry and physical laws in the description of nature.

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Attempts to geometrize physical forces based on this complementarity showed only formal success (Hertz, 1958) until the pseudo-Euclidian metric of the space-time manifold was discovered in the special theory of relativity. Einstein achieved the description of the gravitational field in terms of Riemannian geometry.

The global symmetry of the space-time geometry of special relativity with respect to the Poincaré group is thereby broken and restricted to individual systems of reference in the local limit (see Pauli's version of the principle of equivalence (Pauli, 1958). But even in this local limit, the geometrical formulation does not lend a unique preference to the Poincaré group because the principle of equivalence can also be formulated in terms of a family of other groups, for example the de Sitter group (Halpern, 1977a). Poincaré covariance is, however, still postulated in the local limit for the "alien in the theory" the matter fields on the right-hand side of Einstein's equations.

Attempts to include also the matter fields into the geometry began with Weyl's gauge theory which describes electromagnetism in terms of a non-Riemannian connection (Weyl, 1922). A different approach suggested by Kaluza (1921) and subsequently by Klein (1926) fuses formally the manifold of the electromagnetic gauge group with that of space-time into a five-dimensional pseudo Riemannian space. The metric can be projected on space-time and describes besides the gravitational, also the electromagnetic potentials. The method can be generalized to non-Abelian gauge groups (De Witt, 1964).

This successful (although rather formal) geometrization raises the question how Riemann's complementarity of geometry and physics may affect the uniqueness of the invariance group of a physical theory. We have already pointed out that the Poincaré group enters into the geometrical part of general relativity only in a limit and is even there not uniquely determined (Halpern, 1983a). We may consider it as a rather arbitrary construction: if the group does not apply rigorously (and this is the case whenever the performance of a measurement perturbs the symmetry) we are postulating the existence of culprits responsible for the symmetry breaking (and expect that even these will fulfill the law of the group). This is not a unique procedure. We may, however, be able to make one choice which allows a much more economic description than others (as the Poincaré group and Einsteinian physics proved to do compared to the group Galilei and Lenard's "German physics," which required always new physical *dei ex machina*).

To illustrate the situation, let us bring the principle of inertia into a modernized group theoretical from: *A body moves along a timelike orbit of the translation group in Minkowski space unless forces act on it.*

We compare this with an alternative version: *A body moves along a timelike orbit of the de Sitter group in the de Sitter universe unless forces act on it.*

The de Sitter universes are generalized four-dimensional spherical surfaces immersed into a five-dimensional manifold with one or two imaginary coordinates; the first is spatially closed and open in the timelike directions, the second is open in spacelike directions, but closed in the timelike directions. The corresponding de Sitter groups $SO(4,1)$ and $SO(3,2)$ are the groups of motions on these surfaces. If the radius of the sphere is very large, one may locally not be able to distinguish them from the Poincaré group; but the mathematical structure of these simple groups differs fundamentally from that of the Poincaré group, which is not even semisimple.

A priori considerations such as the interrelation of all natural phenomena and the beauty of the mathematical structure would definitely lend preference to the simple group.

Both versions of the principle of inertia admit every timelike geodesic as a trajectory. The de Sitter covariant version admits, however, in addition a six-parameter family of nongeodesic one-dimensional orbits. These cannot be simply discarded without violating the spirit of the principle, because we are dealing here with a simple group where all generators are interrelated.

The equations of the one-dimensional orbits are (Halpern, 1980a, 1982a)

$$\ddot{x}^k + \left\{ {k \atop he} \right\} \dot{x}^h \dot{x}^e = \frac{1}{2} R^k{}_{hab} S^{ab} \dot{x}^h$$

$$\frac{D}{ds} S^{ab} = S^{ab}{}_{;k} \dot{x}^k = 0, \qquad S^{ab} = -S^{ba} \tag{1}$$

($R^k{}_{hab}$ the Riemann tensor)

If $S^{ab} \neq 0$ initially, the orbit is in general not geodesic. The equations fulfill, however, the more general conditions that have been deduced by Mathisson (1937) and Papapetrou (1951) from the conservation laws for the motion of a test particle with dipole structure (inner angular momentum) in general relativity.

This raises the question as to whether a theory based on local de Sitter covariance yields necessarily a less economic description. After all, the foregoing promises that it may describe the mysterious phenomenon of inner angular momentum on an equal footing with conventional motion. The task is in any case tempting to construct a modification of general relativity rigorously based on local covariance with respect to a group that is at least semisimple. This should make a difference to local Poincaré covariance, namely, if the task is performed rigorously enough that the structure of the semisimple group comes into play. Most discussions on the subject try to avoid just that by insisting that the "respectable" Poincaré group physics be obtainable in the limit when the radius of the universe tends to be infinite. This reduces the introduction of the de Sitter group (and other alternative groups) to an artifice for the elegant formulations of conventional physics. The first attempt for a modification of general relativity based on the de Sitter group is probably due to Lubkin (1971). This work stresses already the purpose of a modified symmetry. The present author adopted early the point of view that the introduction of a group symmetry obliges to search for all of its consequences. The absence of some orbits is not admissible in case of a simple invariance group without further justification. It was speculated that these orbits may be unobservable in laboratory dimensions for statistical reasons and manifest themselves in the very large —or as a kind of virtual motion in the small to produce the phenomenon of elementary particle spin (Halpern, 1979a). Taking the group covariance seriously, led to the construction of a mathematical model which is based so completely on group theory that it was entitled "At the Beginning, there was the Group." The resulting physical theory (Halpern, 1979b, 1980b) is probably more than any other dictated by a mathematical structure (group geometry)—not, however, by mathematicians. Technicalities can hardly be avoided to introduce it,[1] but the results can be presented in the conventional form of a Yang–Mills theory in curved space.

The group G with semisimple Lie subgroup H acts on the factor space G/H. The natural projection π: $G \to G/H$ defines geometrical objects on the factor space in terms of those of the group manifold. A metric g is thus defined on G/H by the Cartan–Killing metric γ on G. The metric of the de Sitter universe results if G is $SO(3,2)$ or $SO(4,1)$ and H is $SO(3,1)$. $P(G, \pi, G/H, H)$ forms a principal fibre bundle and γ defines a horizontal vector space which is perpendicular to the vertical vectors.

The projections of the one dimensional group orbits on G (which are geodesics) onto the base manifold $B = G/H$ are candidates for the equations of motion of a test particle. One has to generalize the exclusion of spacelike trajectories to eliminate those orbits which neither describe structureless nor spinning test particles.

The construction outlined derives topology, metric, and the modified law of inertia from group geometry alone. The symmetry breaking due to

[1]Most of the mathematical apparatus was presented at this meeting in the lecture of Dr. Bleecker.

localized mass distributions has then been introduced by the author in the following way (Halpern, 1979b, 1980b, 1980c): The metric γ of G can be shown to fulfill homogeneous Einstein equations in ($r = \dim G$) dimensions with a cosmological member. A localized mass distribution should give then rise to a right-hand member of these equations. Admissible solutions for the generalized metric γ must retain the topological properties of the fibre bundle $P(G, \pi, G/H, H)$ and thus have Killing vectors with the commutation relations of the generators of H to ensure the projection π. One arrives this way at a multidimensional Kaluza–Klein theory with non-Abelian gauge group (Halpern, 1979b, 1980b). Generalized theories of this kind have been considered before, (De Witt, 1963) but this seems to have been the first case where such a theory in r dimensions is derived from a generalization of a Lie group G_r. Previous work which starts out from a group manifold G_r (Neeman and Regge, 1978) attempts agreement to conventional Poincaré group physics by "softening" the conditions imposed by the group geometry before these do affect the physics as stringently as in the present case. The approach can then also not lead consistently to a Kaluza–Klein type of theory which was adopted in this context only later by rather artificial means.

The author had adopted the view that once determined on a particular invariance group, mathematics should dictate the physics, irrespective of how fashionable and even unrealistic the outcome may appear. The above construction was compared with a fairy tale (Halpern, 1982b) which has a pre-established outlook and features which seem to be irreconcilable with reality yet which at a closer contemplation yield a deeper insight than a profane realistic view.

The opinion is rather fashionable that it is a virtue of general relativity to admit for its solutions a choice of topologies. A physical verification for this assertion can hardly be supplied. *A priori* it appears equally likely that the universe admits only one topology as is the case in the present theory.

The Killing vectors of the Kaluza–Klein theory restrict the solutions of Einsteins's equations in r dimensions so that in four dimensions they are equivalent to a metric and $r-4$ Yang–Mills fields. Further restrictions on the sources and the initial conditions can be imposed so that the connection on P is Riemannian and thus torsion free (Halpern, 1983b, a). The theory in this case becomes much more transparent. It is a metric theory in four dimensions. The field equations are derived from a Lagrangian which contains also terms which are quadratic in the Riemann tensor. They admit the vacuum solutions of general relativity with cosmological member. The resulting equations of motion of test particles due to the nonlinear term are generalized from geodesics to the form of equations (1). Particles which describe the generalized motion interact with the metric-like particles with spin structure in general relativity. This is demonstrated by the energy momentum tensor of a field which is structureless in $r = 10$ dimensions. In four dimensions this yields an energy-momentum tensor plus six Yang–Mills currents. The Riemannian connection allows to combine all these into one energy-momentum tensor which depends, however, on the tetrades, thus giving evidence of a spin structure—even of a Fermion structure.

The "charges" of elementary particles of a Yang–Mills theory can be related to the elements of the fundamental group and thus assume only discrete values if H is multiply connected. A mysterious feature of all Kaluza–Klein theories has hitherto been the gigantic ratio between these charges and the gravitational charges of an elementary particle. The justification that the extensions of the manifold of H have to be so microscopic to make it unobservable, appears not very convincing, as the homogenuity of the space should achieve that. Staying faithfully to our mathematical guidelines rewards us in the case of $SO(3,2)$ with a fascinating outlook: The world line of a test particle with suitable spin (Fermion) occurs on the time-closed de Sitter universe at a given time not just once, but with a large number of appearances at different space points. It describes accordingly as many fermions of the same type. The extension of H is related to the total of all charges and needs no artificial reduction.

The idea that all fermions of one type are due to a single world line has been suggested by Wheeler and its inadequacies led to Feynman's positron theory (Feynman, 1965). It emerges here naturally from the group structure.

Suitable units for our picture make the velocity of light as well as the gravitational constant G dimensionless and one. The unit of length is the radius of the universe R. This makes Planck's constant of dimension of length squared and smaller than 10^{-120}. The work was started in the hope that application of the formalism to $SO(4,2)$ and projective relation of the base manifold will yield a generalization of Jordan's theory (Jordan, 1955) unifying electromagnetisms and gravitation in accord with Dirac's present form of the large number hypothesis (Dirac, 1979).

The limitation of the theory to vanishing torsion appears artificial, reminding somewhat of the restricted conformal metric theory of Einstein and Fokker (Einstein and Fokker, 1914). The presence of torsion generalizes the curvature tensor in equations (1) and the nonlinear part of the field equations. It results in a modified spin–spin interaction of the sources due to additional degrees of freedom of the fields. These can only be calculated if the interaction of the tetrads with the matter fields are known. This is in general not the case for microscopic averaged description of sources as it occurs in general relativity. Every macroscopic rotating body is subject to binding forces of fields with spin. Torsion is here not localized and identified with spin as in the Einstein–Cartan theory and its newer versions

(Cartan, 1923; Hehl, 1980). Torsion is here associated to independent degrees of freedom and propagating. It has to be assumed that it interacts universally with all angular momenta. The relation to Mach's principle is thereby deeply modified in this theory. The present work contains only a preliminary study of the properties of the source term. Special caution has to be applied also to the introduction of quantum laws because one has to expect deep-going modifications.

A consistent description of spin in terms of representations of the Poincaré group exists in the conventional theory (Wigner, 1939; Laurent, 1963) and one may (somewhat artificially) even bring a Poincaré covariant theory into a similar form as in our theory. The relation to the metric is then much less convincing and the mathematical beauty is impaired. We prefer to make full use of our right based on Riemann's complementarity.

Why do we renounce on the acclaim that could be collected by creating a super symmetric generalization of our theory? We believe that mathematical sophistication leads to physical progress mainly in the cases where we either cannot manage to perform our calculations within a simpler mathematical apparatus or when clear and detailed physical concepts are missing. It can easily prove misleading in other cases. Suppose Balmer's formula had not been discovered. The spectral problem would today no doubt be attacked with the most sophisticated means of analysis. Would this help or hinder the finding of Bohr's quantum rule? Many mathematicians admittedly feel attracted to spaces with supersymmetric structure only because of their mathematical sophistication. A main task of the physicist different from the mathematician is to separate from a diverging number of mathematical possibilities the one best adapted to the phenomena. The formulation of a diverging[2] number of theories even if so general, that probably some natural phenomena are approximated occasionally by them, can be of negative merit to physics.

It can of course, not yet be claimed that the concepts of the present theory are physical, but they are clear as a model and they suggest no other symmetry between fermions and bosons than that arising naturally by the composition of bosons from fermions. The symmetry suggested by Golphand and Likhtman (1971) does not occur naturally here. We do not have yet to struggle with renormalization problems and have thus no reason to introduce a supersymmetric version. It is hoped that the structure that seems to introduce spin will itself teach us also more about its statistical laws.

[2]People who remember how fashion creations started in the late fifties to compete in accentuating curves and symmetries that easily catch our attention, going far beyond that which can be expected in nature, could by analogy be drawn to the expectation that even in physics, superfashion may finally give way to a new miniline.

2. THE GROUP MANIFOLD AND ITS GEOMETRY

The theory in its final four-dimensional form can be expressed in the formalism of classical general relativistic field theory. The fundamentals of the differential geometry of Lie groups are, however, a prerequisite for an understanding of its structure. The use of theorems from the theory of fiber bundles allows an elegant global presentation. The reader does not really need these for the examples considered because their bundles are trivial.

We consider an r-dimensional Lie group G which acts transitively on a k-dimensional manifold B as a group of transformations. Then in general, G has an $(r-k)$-dimensional Lie subgroup H which leaves one element of B invariant. B is then homeomorphic to the coset space G/H and a natural map π from G onto B is defined. (The group manifold of G is the bundle space of the principal fiber bundle $P(G,\pi,H,B)$ with base B and group and typical fiber H) (Steenrod, 1974; Nomizu, 1956).

We keep here always the example of the de Sitter groups, especially $SO(3,2)$ for G and the Lorentz group $SO(3,1)$ for H in mind. B can then be identified with the manifold of the de Sitter universe.

The multiplication of group elements: $c=a.b$ in a local chart reads

$$c^t=\phi^t(a,b)=\phi^t(a^1\cdots a^r,b^1\cdots b^r) \tag{2}$$

forming the differential:

$$dc^t=V_u^t(a,b)\,da^u+W^t(a,b)_u\,db^u \tag{2a}$$

associativity: $\phi(a,hb)=\phi(ah,b)$ implies that the r one-forms (covariant vectors)

$$A^T(x)=W^t(x^{-1},x)_u\,dx^u \tag{3}$$

are invariant with respect to left translations $G\times G\to G$: $L_a x=a.x$ and the r one-forms

$$\bar{A}^T(x)=V_u^t(x,x^{-1})\,dx^u \tag{3a}$$

are right invariant.

A left-invariant two-form (covariant antisymmetric tensor) can be expanded in terms of the exterior product of pairs of left-invariant one-forms with constant coefficients. L_a^* (the pullback of L_a acting on forms) commutes with exterior products and the exterior derivative d so that a relation

$$dA^T + (1/2)c_U{}^T{}_V A^U \wedge A^V = 0 \tag{4}$$

is fulfilled for constant $c_U{}^T{}_V = -c_V{}^T{}_U$ (structure constants). In components with commas denoting derivatives:

$$A^T{}_{t,s} - A^T{}_{s,t} + (1/2)c_U{}^T{}_V\left(A^U{}_s A^V{}_t - A^U{}_t A^V{}_s\right) = 0 \tag{4'}$$

the right-invariant forms fulfill the corresponding equations

$$d\bar{A}^T - (1/2)c_U{}^T{}_V \bar{A}^U \wedge \bar{A}^V = 0 \tag{4a}$$

with the same structure constants (Maurer–Cartan equations).

The r left-invariant vector fields A_S dual to the forms A^T: $A^T(A_S) = \delta^T{}_S$ form a base to the tangent space of the manifold. They fulfill

$$[A_U, A_V] = c_U{}^S{}_V A_S \tag{4b}$$

The right-invariant vectors dual to the $\bar{A}^T$ fulfill correspondingly:

$$\left[\bar{A}_U, \bar{A}_V\right] = -c_U{}^S{}_V \bar{A}_S \tag{4c}$$

$$\left[A_U, \bar{A}_V\right] = 0 \tag{4d}$$

A Lie group is simple if it has no proper invariant subgroup, and semisimple if it has no Abelian invariant subgroup. The components of the Cartan–Killing metric γ on the manifold of a semisimple group G are in an unholonomic frame (Eisenhart, 1933):

$$\gamma_{ST} = c_S{}^U{}_V c_T{}^V{}_U \tag{5}$$

and in local coordinates:

$$\gamma_{uv} = A^S{}_u \gamma_{ST} A^T{}_v \tag{5a}$$

The Ricci tensor of this metric on G fulfills the relation:

$$R_{uv} = 1/4\gamma_{uv} \tag{6}$$

The author has reinterpreted this relation into:

$$R_{uv} - 1/2\gamma_{uv}R + 1/8(r-2)\gamma_{uv} = 0 \tag{7}$$

that means the metric fulfills homogenous Einstein equations in r dimensions with a cosmological member (Halpern 1979b, 1980b, c). The author showed that the metric γ fulfills all the conditions of a special solution of a peculiar form of a Kaluza–Klein theory in r dimensions with gauge group H (Halpern, 1979b, 1980b, c).

The (group) manifold P is locally homeomorphic to $B \times H$. The mapping $P \to H$ depends at each point of B on the particular local trivialization chosen.

We shall frequently use local charts in such trivilizations and adopt our notation as follows:

All indices (capital for unholonomic bases and small for (coordinates) running from 1 to K, by letters in the alphabet before L. Indices running from $K+1$ to R, by letters from L to inclusive Q. Indices running from 1 to R (dimension of group manifold) are denoted by letters after Q in the alphabet. The summation conventions are also adjusted to this rule. Thus, $A_h B^h$ or $A_H B^H$ implies the summation from 1 to K (dimension of base manifold). $A_m B^m$ implies summation from $K+1$ to R and $A_s B^s$ from 1 to R. We shall use this convention without further warning.

A base of left-invariant vectors is always so chosen that the last $r-k$ vectors A_M belong to H. A local trivialization admits always coordinate systems in which the components of these $A^t{}_M$ do not depend on the k coordinates x^i and all their x^i components also vanish (Eisenhart, 1933).

B is the space of left cosets of H. Right translations give rise to coadjoint transformations of the left-invariant forms (and adjoint transformations of left-invariant vectors):

$$L^*_{a^{-1}} R^*_a A^S(x) = \bar{A}^S_t(a) A^t{}_V(a) A^V(x) = ad(a^{-1}) A^S(x) \qquad \text{(4e)}$$

and correspondingly

$$R^*_{a^{-1}} L^*{}_a \bar{A}^S(x) = A^S_t(a) \bar{A}^t{}_V(a) \bar{A}^V(x) = ad(a) \bar{A}^S(x) \qquad \text{(4e)}$$

It follows from the group property that the left-invariant base vectors A_S and right-invariant $\bar{A}_S$ are expressible as

$$A^t{}_S(x) = W^t(x, e)_s, \qquad \bar{A}^t_S(x) = V^t_s(e, x) \qquad \text{(4f)}$$

Equations (2, 2a) shows that they are the generators of right and left translations. Equations (4b)–(4d) give the infinitesimal adjoint tranformations of the base vectors.

Adjoint transformations leave the structure constants and thus the metric γ invariant[3]:

$$[A_s, \gamma] = 0$$

[3]The Lie bracket is used in typescript for all Lie derivatives.

Applied to $S = M > K$ this implies that a projection of the contravariant metric tensor of γ from P onto B by the differential of π is independent of the x^m and thus uniquely defines a metric g on B. One can infer from equation (5a) that the orbits of one-dimensional subgroups are geodesics on P.

The frame vectors A_S can be chosen orthogonal to each other, with respect to the metric γ. A horizontal vector space perpendicular to the vertical vector space spanned by the A_M is thus defined; it is spanned by the A_E.

A connection form ω is given by

$$\omega = A^M(x)\hat{A}_M, \qquad \omega(A_E) = 0 \tag{8}$$

with $\hat{A}_M$ the element of the Lie algebra pertaining to A_M. If, as in our examples, all structure constants of the form $C_M{}^P{}_E$ vanish, the required transformation properties of ω follow from equation (4e).

The curvature two form Ω of ω is give by Cartan's structure equations:

$$\Omega = d\omega + [\omega, \omega] \tag{9}$$

its coordinate components are

$$\Omega_{ik} = F_{ik}^M \hat{A}_M = \left(A_{k,i}^M - A_{i,k}^M + A_i^P c_P{}^M{}_Q A^Q{}_k\right)\hat{A}_M \tag{9a}$$

C_{PQ}^M belong to H only so that this expression does not vanish when equation (4a) is fulfilled. On the group manifold one has thus

$$F_{ik}^M = -c_E{}^M{}_F A_i^E A_k^F \tag{9b}$$

The pseudo-orthogonal group H is a restriction of the general linear group, so that $A^M c_M{}^E{}_F$ can be related to a linear connection on the frame bundle with a soldering form: $\theta = A^E \hat{e}_E$, where the $\hat{e}_E$ form a vector base on $\mathbb{R}^k$. The torsion form θ is defined in terms of the (vector)-values which it assumes for two vectors U, v:

$$\theta(u,v) = d\theta(u,v) + \omega(u)\theta(v) - \omega(v)\theta(u) \tag{10}$$

in coordinate components for our case:

$$\theta_{ik} = F_{ik}^E \hat{e}_E = \left(A_{k,i}^E - A_{i,k}^E + c_M{}^E{}_F\left(A_i^M A_k^F - A_k^M A_i^F\right)\right)\hat{e}_E \tag{10a}$$

The Maurer–Cartan equations (4) for $T = E$ show that the construction is torsion free. The (pseudo-) orthogonal H implies the vanishing of the covariant derivative *of the metric* so that on the group manifold the connection is Riemannian.

The left-invariant vector field of a horizontal generator of G is horizontal on the whole group manifold. The projection of an orbit of this field is a geodesic on B [$S = 0$ in equation (1)]. This is in our examples the analog of a maximal circle on the generalized sphere. A timelike geodesic on B is closed in the case of $G = SO(3,2)$. If the one-dimensional group orbit on G has also vertical components, its projection on B, if timelike, will in general no more be a circle, but recur repeatedly at different space points for any given time. The number and positions of the recurrences depends on the ratio of the horizontal and vertical components.

3. THE SHADOW PLAY OF PHYSICS ON SPACE-TIME

We have obtained in Section 2 the metric of the de Sitter universe as well as the geodesic trajectories of test particles from the geometry of the group manifold alone. The structure of the simple de Sitter group, which exhibits a maximal symmetry, gives us no justification to exclude other timelike orbits as candidates for particle trajectories. Our aim is to explore the physical implications of an invariance group in all its aspects and we should not introduce arbitrary modifications at this stage. The orbits of all one-dimensional subgroups of a (pseudo-) orthogonal group on the homogeneous space are given by the equations (1). We have already indicated in Section 1 how we want to interpret the nongeodesic orbits. Group geometry also determines the metric and topology of the universe. We attempt just to approximate the description of local inhomogenous matter distributions by a right-hand member of the field equations which we obtained from group geomety and we know that this must result in the above orbits of test particles. In classical general relativity solution of the homogeneous equations are identified with the absence of matter (this is assumed by most people even if one has a cosmological member). We keep meanwhile this assumption although it may later have to be modified.

The general theory of relativity requires boundary conditions for the field equations. The present theory provides the cosmological boundary conditions and further conditions for the solutions of the r-dimensional field equations from group geometry.

The generalized solution $\bar{G}$ must retain the properties of a bundle space of the same principal fiber bundle $P(\bar{G}, \pi, H, B)$ as before. ($\bar{G}$, B have the same topology as in case of the group manifold and H forms still the structure group and typical fiber). The metric γ is generalized, but $r - k$

vector fields A_M must still exist on $\bar{G}$ that are Killing vectors [fulfill equation (5b)] and obey the commutation laws of generators of H. γ still determines a horizontal vector space which is perpendicular to the vertical A_M. The existence of a (Lie-algebra-valued) connection form ω and a (R^k-valued) soldering form θ is met by requiring the existence of k horizontal orthonormal vector fields B_E which fulfill

$$[A_M, B_E] = C_M{}^F{}_E B_F \tag{4b'}$$

with the structure constants of G. Only the commutation rules between these k vector fields remain undetermined.

To give ω the form of equation (8) the $(r-k)$ one-forms A^M are modified to fulfill: $A^M(A_N) = \delta^M_N$ and $A^M(B_E) = 0$. They still fulfill equation (4e) for $a \in H$.

θ can be expressed with the k one-forms B^E, fulfilling $B^E(A_M) = 0$ $B^E(B_F) = \delta^E_F$, as $\theta = B^E \hat{e}_E$. Thus equations (4e) and (4b′) guarantee the correct transformation properties of the forms with respect to the right action of H on P. The curvature two-form Ω is still given by equation (9) and the torsion form θ by equation (10) expressed in terms of the B^E. The projected metric $g = \pi'\gamma$ on B is now generalized. Its components are

$$g^{ik} = B^i_E \gamma^{EF} B^k_F, \qquad \gamma_{ik} = B_i^E \gamma_{EF} B_k^F + A_i^M \gamma_{MN} A^N{}_k \tag{11}$$

The restrictions imposed on the solutions γ of the generalized field equations imply that for any local trivialization the Lagrangian for the left-hand member of the field equations which depends on the metric γ, is equivalent ($\cong$) to a Lagrangian on the base manifold depending on the metric g and the connection form. Choosing coordinates on the fiber so that $\det(A^m_M) = 1$, we obtain

$$\begin{aligned}\mathscr{L}_G^{(r)}(\gamma) &= \sqrt{\gamma}\,(R^{(r)} + \Lambda) \cong \mathscr{L}_G^{(k)}(g, A^M) \\ &= \sqrt{g}\left(R^{(k)} + 1/4\gamma_{MN} F^M_{ij} F^{Nij} + \lambda\right)\end{aligned} \tag{12}$$

For $r = 10$ we have $\Lambda = -2$ and $\lambda = 1/2$; the difference arises from the curvature invariant on the group manifold of H. Other formulations of Kaluza–Klein theories choose a flat metric for this manifold (Kerner, 1968). The two de Sitter universes have here the same λ, but metrics of opposite signature. The Lagrangian in the four-dimensional form is that of a gravitational field plus the Yang–Mills field F^M.

The connection coefficients of our linear connection are in an unholonomic frame $\{B_E, A_M\}$ given by

$$\Gamma_H{}^E{}_F = B^i_H A_i^M c_M{}^E{}_F \tag{13}$$

We can with the help of equation (10a) express this in terms of the Riemannian connection and the torsion tensor $F^{e}{}_{.hk}$ in a natural frame:

$$\Gamma_{hk}^{e} = \left\{ {e \atop hk} \right\} + 1/2 (F^{e}_{.hk} + F_{k\ h}^{.e} + F_{h\ k}^{.e})$$

$$F^{e}_{.hk} = B^{e}_{E} F^{E}_{hk} \tag{13a}$$

The covariant derivative of g with this connection vanishes.

$$\nabla_{h} g_{ab} = g_{ab,h} - \Gamma^{e}_{hb} g_{ae} - \Gamma^{e}_{ha} g_{eb} = 0 \tag{13b}$$

The curvature tensor $F^{i}{}_{jhk}$ in a natural frame becomes according to equations (9)

$$F^{i}{}_{jhk} = \Gamma^{i}_{kj,h} - \Gamma^{i}_{hj,k} + \Gamma^{i}_{ha} \Gamma^{a}_{kj} - \Gamma^{i}_{ka} \Gamma^{a}_{hj}$$

$$F^{i}{}_{jhk} = F^{M}_{hk} c_{M}{}^{J}{}_{I} A^{i}_{J} A^{I}_{j} \tag{13c}$$

so that the Lagrangian can be expressed as

$$\mathscr{L}_{G}(g, \Gamma) = \sqrt{g} \left(R + \lambda + 1/48 F_{ijhk} F^{ijhk} \right) \tag{12a}$$

The Lagrangian density is covariant with respect to general transformations of the coordinates x^{r}. The transformations generated by a change of local trivializations are of special interest. They are of the form

$$x'^{k} = x^{k}, \qquad x'^{m} = \phi^{m}(x^{h}, a^{n}(x^{h})) \tag{2b}$$

with $a(x^{h}) \in H$ depending on the points of the base. The coordinates of the points of the base remain unchanged. Comparison of the components of the forms A^{M}, B^{E} at points which have also the same vertical coordinates shows with the help of equations (4b′,e) that they are related by an adjoint transformation. Only the components A^{M}_{i} acquire in addition an inhomogenous term:

$$A'^{M}_{r}(x) = ad(a) A^{M}_{r} - A^{M}_{n}(a) a^{n}(x^{h})_{,r} \tag{2b′}$$

The dual vectors are transformed accordingly. The B^{i}_{E} in particular, undergo a tetrad rotation; only the B^{m}_{E} have an inhomogeneous term. In the $(k = 4)$-dimensional form of the theory the coordinate transformation associated with the change of local trivialization manifests itself as a gauge transformation of the potentials $A_i^M(x^h)$ and fields $F_{ik}^M(x^h)$, combined with a rotation of the tetrads B_E^i. The fields F^R according to (2b′), undergo a homogeneous adjoint transformation.

Inhomogeneous localized matter distributions are described by a right-handmember $\sqrt{\gamma}\, T_M^{rs}$ of the field equations. It is derived from a matter term $\mathscr{L}_M$ of the Lagrangian which depends on γ and the matter fields:

$$\sqrt{\gamma}\, T_M^{uv(r)} = -\frac{1}{2}\left(\frac{\delta}{\delta\delta_{uv}} + \frac{\delta}{\delta\delta_{vu}}\right)\mathscr{L}_M \tag{12a'}$$

The Kaluza–Klein theory requires for every Lagrangian the conditions:

$$[A_M, \mathscr{L}] = 0 \tag{12b}$$

Together with equation (5b) this requires for the energy-momentum tensor

$$[A_M, T_M] = 0 \tag{12c}$$

besides the conservation law.

The geodesics of the metric γ on $\bar{G}$ are solutions of the equations of motion resulting from our field equations (Papapetrou, 1951). If we ascribe to them the orbit of a test particle, we must keep in mind that its matter distribution does in general not fulfill equation (12c) even in the limit. The description of the motion is thus only an approximation. We consider nevertheless first the projection of these trajectories on B, although they are complicated by the tetrads, the orientation of which depends on the points of the fibre cut by the geodesic. The general equations for the projection of geodesics on B are

$$\ddot{x}^k + \left\{ {k \atop he} \right\} \dot{x}^h \dot{x}^e = 1/2 F^k{}_{hab} \dot{x}^h S^{ab}$$

$$\frac{D}{ds} S^{ab} = S^{ab}{}_{;k} \dot{x}^k = 1/2 \dot{x}^k S^{hb} (F^a_{.hk} - F_{k\ h}^{\cdot a} - F_{hk}^{\cdot\cdot a}) - (a \leftrightarrow b)$$

$$S^{ab} = -S^{ba} \tag{1a}$$

The "spin tensor" S^{ab} vanishes if the initial direction of the geodesic is horizontal. If the torsion tensor $F^a_{.bc}$ vanishes, we have equation (1) with the Riemann tensor $R^k{}_{hab}$ on the right-hand side. These equations describe also the orbits of a whole family of semisimple groups on their homogeneous spaces. The right-hand side in case of our pseudo-orthogonal group assumes the simple form: $S^k{}_h \dot{x}^h$. The projection of timelike horizontal geodesics is geodesic and closed in this case. The equations (1) describe thus the trajectories also in case of a general metric g if torsion vanishes. They are then still in accord with Papapetrou's conditions for the motion of spinning test particles (Papapetrou, 1951). These conditions are necessary, but hardly sufficient for the description of the particle. One notices that in case of more complicated curvatures $S^{i}{}_{k}\dot{x}^{k}$ cannot remain zero. (Actually this is the more interesting situation we encountered.) This poses no problem in curved space, but it persists in a flat space region. This may mean that in the rest system the center of gravity does not coincide with the coordinate of the particle. A more detailed study of an energy-momentum tensor T_M which does fulfill equation (12c) has to be made in order to see how the theory describes spin. We first limit the contemplation to the case of vanishing torsion. In $k = 4$ dimensions the left-hand member is derived from the Einstein Lagrangian plus a cosmoslogical term quadratic in the curvature plus a cosmological constant λ. This yields all vacuum solutions of general relativity with λ and no doubt additional solutions about which little is known.

We contemplate now the right-hand member first in all r dimensions. There occurs the energy-momentum tensor $T_M^{(r)}$ as source term. We assume that this term is structureless in r dimensions (e.g., that of a scalar field interacting with γ)[4]. We choose again a coordinate system x^m on the fibres in which $\det(A_n^M) = \text{const}$. The r-dimensional formulation, because of conditions (5b) and (12c), is again equivalent to a k-dimensional one with $\mathscr{L}_M^{(4)}$ depending besides of the matter field on g and A_i^M. Defining $T_M^{ik(4)}$ by the variation with respect to g_{ik} in analogy to equation (12a′) and the current

$$j_N^k = \frac{-\delta\mathscr{L}_M}{\delta A_k^N}, \quad j^{k[a,b]} = j_N^k \gamma^{NM} c_M{}^E{}_F \gamma^{FH} A_E^a A_H^b \tag{14}$$

one finds that the conservation laws $\sqrt{\gamma}\, T^{st}_{M;t} = 0$ can in $k = 4$ dimensions be expressed as

$$\left(\sqrt{g}\, T^{ik(4)}\right)_{;k} = F^{iN}_{h} j_N^h \tag{15a}$$

$$j^k_{N,k} = c_P{}^Q{}_N j_Q^k A_k^P \tag{15b}$$

The components A_k^P depend on the point on the fibre and they occur thus naturally in conjunction with a preferred system of tetrads B_i^E. A gauge transformation affects both of them. [If torsion vanishes equation (13)

[4]The scalar field may have several components (a complex field has two components). It is a vector in the Hilbert space of square integrable functions of x^m.

identifies $A_k^M c_M{}^E{}_F B_H^k$ with the Ricci rotation coefficients which are expressible in terms of the B_E (Eisenhart, 1932).] The result of the r-dimensional theory can be expressed on the base. The matter field (which is structureless in r dimensions) due to its x^m dependence, projects on the base with an inner structure. This structure should be a representation of H if we want to describe a boson field in the simplest way. The coordinate frame for the connection can then be used. To describe fermions we have the choice either to use the covering groups of G and H (and compromise the spin quantization mechanism outlined) or expand on the remarkable new possibility suggested in the introduction. Equation (12c) remains valid even for spinnors which are a representation of the covering group of $SO(3,1)$ rather than of this group itself. The tetrads are in this case required to "bridge" the representation to the coordinates.

The condition which we have posed, that the tensor T_M be in accord with equation (12c) cannot be fulfilled with a test particle which is point like on $\bar{G}$. The tetrads in this case can therefore not be eliminated so that the difficulties mentioned previously for the interpretation of the orbits in special cases result. It served us mainly for heuristic purposes.

We consider now the general case where the coordinate frame of the tetrads is not used. We combine $T_M^{ik(4)}$ and j^k to a total matter tensor τ on the base by setting

$$\sqrt{g}\,\tau^a{}_b = B_b^E \delta \mathscr{L}^{(k)} / \delta B_a^E \tag{16}$$

After a lengthy calculation one finds

$$\sqrt{g}\,\tau^{ab} = \sqrt{\gamma}\, T_M{}^{ab(r)} + 12\left(j^{k[b,a]} + j^{b[k,a]} + j^{a[k,b]}\right)_{:k} \tag{17}$$

The form of the term in the parantheses is similar to that found by Rosenfeld (1940) for the symmetrization of the canonical tensor. τ is conserved if the field equations are satisfied.

The restriction to vanishing torsion results thus in a metric theory in which spin of matter fields appears in the conventional form (without having been inserted). Spin–spin interaction occurs only via the metric. The theory differs besides this from Einstein's theory with cosmological member on the classical level by a large cosmological term quadratic in the Riemann tensor. Such a term occurs also in other gauge theories of gravitation related to spin rotations (Halpern, 1977b).

The restriction to vanishing torsion appears artificial. The inclusion of torsion leads to a new theory with an additional spin–spin interaction besides that caused by the metric. It is obtained by admitting a wider class of solutions of the r-dimensional metric theory. In $k = 4$ dimensions the

Lagrangians $\mathscr{L}_G$ and $\mathscr{L}_M$ are varied *independently* with respect to the metric g (or the tetrads B_E) and the connection. Variation with respect to the metric yields the Einstein tensor with a cosmological member and an additional right-hand member apart from T_M. This additional source term should not be very important in the case of weak fields. The variation with respect to the connection yields nonlinear equations of second order of the Yang–Mills type with the currents of equation (14) as sources. The torsion can then be calculated from the connection according to equation (13a). Notice that even the vanishing of the connection τ, but not of the Christoffel connection $\{{}_{j}{}^{i}{}_{h}\}$ can result in nonvanishing torsion. We must impose to all solutions due to localized sources, the cosmological boundary condition of vanishing torsion and de Sitter geometry in the limit of large spatial distances:

$$\sqrt{g}\,\big[R_{ik}-1/2g_{ik}(R+\lambda)+F_{iabc}F_{k}{}^{abc}+F_{aibc}F^{a}{}_{kbc}$$
$$+2F_{abci}F^{abc}{}_{k}-1/2g_{ik}F_{abcd}F^{abcd}+T_{ik}{}^{M}\big]=0 \tag{18}$$

and for $\mathscr{L}_M$ which do not depend explicitly on the tetrads:

$$\nabla_k\left(\sqrt{g}\,F_a{}^{bck}\right)-1/2\sqrt{g}\left(F_a{}^{bhk}\Gamma^{c}_{kh}+2F_a{}^{bch}F^{k}{}_{.kh}\right)+j_a{}^{bc}=0 \tag{19}$$
$$j_a{}^{bc}=\frac{\delta\mathscr{L}_M}{\delta\Gamma^{a}_{cb}}$$

These equations apply to a fine grained theory in which the spin of the matter field appears in $k=4$ dimensions explicitly. The structure of a macroscopic spinning particle in a coarse-grained theory may be modeled after this case.

We consider the theory only as consistent if also a system orbiting under the influence of binding forces interacts with Γ in a similar manner. We consider the gravitational field providing the binding force. It is the only field we treated here in detail [the electromagnetic field we hope to be able to describe by a metric in a $SO(4,2)$ covariant theory].

We consider the periodic orbiting of two structureless particles due to their gravitational attraction. Γ and g have in this case to be such that at a larger distance from the system they are of similar form to that of a spinning system. This has to be achieved by the boundary conditions imposed.

Even the two-body problem in the general theory of relativity is far from being solved; it may need much further study to prove in general the physical consistency of the coupling of torsion to spin and orbital angular momenta. This is a condition for the exploration of such global aspects as Mach's principle in the present theory.

Torsion and metric disturbances in an isolated system are coupled to matter by the same constant and should thus be roughly of comparable magnitude.

Experimental verifications can be hoped from the study of elementary photon gyroscopes passing the neighborhood of such large spinning masses as the neutron stars of pulsars. The electromagnetic interaction promises, however, to be better understood when incorporated into a similar unified metric theory with the group $SO(4,2)$. The Stanford gyroscope experiment is expected to be of sufficient sensitivity to detect the effects of torsion considered here.

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REFERENCES

Cartan, E. (1923). *Ann. Ec. Norm. Sup.*, **40**, 325.

De Witt, B. (1964). (in Relativity, Groups and Topology Les Houches Course), Gordon & Breach NY.

De Witt, B. (1963). Les Houches Lecture.

Dirac, P. A. M. (1979). *Proc. R. Soc. London Ser. A*, **365**, 19.

Ebner, D. (1982). *J. Gen. Rel. Grav.*, **14**, 1001.

Einstein, A. and Fokker (1914). *Ann Phys. (Leipzig)*, **44**, 321.

Eisenhart, L. P. (1932). *Riemannian Geometry*. Princeton University Press, Princeton, New Jersey.

Eisenhart, L. P. (1933). *Continuous Groups of Transforms*, Princeton University Press, Princeton, New Jersey.

Feynman, R. P. (1965). Nobel Lecture, Stockholm.

Golphand, Y. and E. Likhtman, (1971). *JETP Letter*, **13**, 432.

Halpern, L. E. (1977a). *J. Gen. Rel. Grav.*, **VIII**(8), 623.

Halpern, L. (1977b). In *Diff. Geom. Meth. In Phys.*, Springer Lecture Notes in Math. No. 570. Springer, New York; *Proc. 1st Marcel Grossman Meeting*, Trieste (N.H.), R. Ruffini, ed.

Halpern, L. E. (1979a). *Int. J. Theor. Phys.*, **18**, 845.

Halpern, L. E. (1979b). *Proceedings of the 2nd Marcel Grossmann Metting*, Trieste, R. Ruffini, ed. North-Holland, Amsterdam.
Halpern, L. E. (1980a). *Int. J. Theor. Phys.*, **20**, 297.
Halpern, L. E. (1980b). In *Physics and Contemp. Needs*, Vol. 5. Plenum Press, New York.
Halpern, L. E. (1980c). Clausthal Summer Institute Preprint.
Halpern, L. E. (1982a). *Int. J. Theor. Phys.*, **21**, 791.
Halpern, L. E. (1982b). In *Gauge Theory and Gravitation, Proc. Symp.*, Nara, Japan, Springer Lecture Notes in Physics, No. 176, Nakanishi, ed. Springer, New York.
Halpern, L. E. (1983a). On complete group covariance without torsion, FSU Preprint, February 1983 (to appear in B. De Witt Birthday Volume).
Halpern, L. E., (1983b). *Found. of Physics*, **13**, 297.
Hehl, F. (1980). *Proceedings of the Erice School on Cosmology and Gravitation.*, Bergmann and DeSabbata, eds. Plenum Press, New York.
Hertz, H. (1958) *Prinzipien Der Mechanik*. Dover, NY.
Jordan, P. (1955). *Schwerkraft u. Weltall.* Vieweg, Braunschweig.
Kaluza, Th. (1921). *Sitzb. Preuss. Akad. Berlin*, 966.
Kerner, R. (1968). *Ann. Inst. Henri Poincare*, **9**, 143.
Klein, O. (1926). *Z. Phys.*, **37**, 895.
Laurent, B. (1963). Elementary particle theory, course at Nordita.
Lubkin, E. (1971). *Symposium on Relativity & Gravitation.*, Kuper and Perez, eds. Gordon and Breach, New York.
Mathisson, M. (1937). *Acta Physica Polonica*, **6**, 167.
Neeman, Y., and Regge, T. (1978). *Rev. N. Cim.*, **1**, N5.
Nomizu, K. (1956). *Lie Groups and Differential Geometry*, Publ. Mathemat. Soc. of Japan 2.
Papapetrou, A. (1951). *Proc. R. Soc. London Ser. A*, **209**, 248.
Pauli, W. (1958). *Theory of Relativity*. Pergamon, London.
Riemann, B. (1876). *Habilitationsschrift* (in his collected works, Goettinger Abh. XIII).
Rosenfeld, L. (1940). *Mem. Acad. R. Belge Sci*, **18**, No. 6.
Steenrod, N. (1974). *The Topology of Fiber Bundles*, Princeton University Press, Princeton, New Jersey.
Weyl, H. (1922). *Space Time Matter.* Methuens, London, New Dover ed.
Wigner, E. (1939). *Ann. Math.*, **40**, 149; *Proc. Am. Acad.*, **34**, 211.